**Spatial Smoothing**

Consider the data \( y_1, \ldots, y_{100} \) representing observed counts of sudden infant death syndrome (SIDS) aggregated from 1979 to 1983 across 100 counties of North Carolina. Typically, such data are assumed to follow a Poisson distribution

\[
y_i \sim \text{Pois}(E_i, e^\mu_i) \quad \text{for} \quad i = 1, \ldots, 100
\]

where \( E_i \) is the expected number of counts and \( e^\mu_i \) is the relative risk for area \( i \). The relative risk is an estimate of the **true standardised incidence rate** (SIR). The raw SIR, \( y_i/E_i \), may be unreliable for two reasons: 1) the observed counts may not be accurate (some SIDS deaths may have gone unrecorded, or recorded as a different cause of death), and 2) the expected counts, which are proportional to the population at risk (live births), may also be inaccurate.

\[
\theta = \frac{y_j}{E_j}
\]

where \( \theta \) is the **relative risk** for area \( j \). In the Bayesian framework, both \( \theta \) and \( \mu \) would be assigned weakly informative priors.

When comparing models, goodness-of-fit (GoF) statistics are usually used to choose the ‘best’ model. These statistics measure model fit while penalising for complexity. As an alternative, we propose goodness-of-smoothing (GoS) statistics which attempt to measure the degree of smoothing. For the methods below, we introduce the **covariate-adjusted SIR** (CASIR) \( e^\beta \), and covariate-adjusted raw SIR (CARSIR) analogous to the raw SIR.

**Goodness-of-Fit Statistics**

The deviance information criterion (DIC)\(^2\) can be expressed as

\[
\text{DIC} = 2p_D - 2 \log \left( \sum_{i=1}^{N} p(y_i|\theta_i) \right)
\]

where \( p_D \) penalises for additional parameters and \( p(y_i|\theta_i) \) is the likelihood evaluated at the posterior mean of \( \mu_i \). The widely applicable information criterion (WAIC)\(^3\) is defined as

\[
\text{WAIC} = 2p_W - 2 \log \left( \sum_{i=1}^{N} p(y_i|\theta_i|y_i) \right)
\]

where \( p_W \) is similarly a penalty term.\(^4\) A third GoF criteria is the leave-one-out conditional predictive ordinate (CPO)\(^5\) which seeks to re-observe theoretical future data \( y_i \) given \( y_{-i} \).

\[
\text{CPO}_i = p(y_i|\theta) = \int p(y_i|\theta)p(\theta|y_{-i})d\theta.
\]

The best models are taken to be those which minimise, \( \text{DIC}, \text{WAIC}, \alpha - \sum_{i=1}^{N} \log(CPO_i) \).

**Goodness-of-Smoothing Statistics**

The **variogram** for area \( i \) at distance lag \( h \) is given by

\[
\hat{\gamma}(h) = \frac{1}{2N(h)} \sum_{i=1}^{N} (y_i - y_j)^2
\]

where \( N(h) \) is the number of areas which are no more distant than the lag \( h \) from area \( i \), and \( j \) denotes all areas \( j \) which satisfy distance \( d_{ij} < h \), and \( z_i \) is a measured spatial variable. This variogram is computed for CASIR and CARSIR, and their ratio is averaged over \( h \). A small ratio (flat variogram) indicates over-smoothing, while a large ratio indicates under-smoothing. An example is shown in Figure 2.

Based on the work of Rong and Bailis (2017)\(^6\), a second approach we propose is to consider models that preserve the **spatial kurtosis** of the SIR,

\[
\text{Kurt}(SIR) = \frac{\mathbb{E}[(SIR - \text{mean})^4]}{\mathbb{E}[(SIR - \text{mean})^2]^2}
\]

while minimising (or reducing) roughness (i.e. the standard deviation of the first-order difference series). We also propose a new method, which considers how far the estimate \( \text{CASIR}_i \) moves from \( \text{CARSIR}_i \) towards the mean of the neighbouring CASIR values, \( \mathbb{E}[(\text{CASIR}_i - y_j)^2] \). Denoting the position at these two extremes as 0 and 1 respectively, the overall degree of smoothing can be quantified by analysing the distribution of these area-specific CASIR positions (see Figure 3).

**Comparing Spatial Models**

Consider the following maps showing a posterior point estimate of the key parameter values and derived quantities for 8 different models. According to DIC, WAIC, and the CPO criterion, the best models are H, G, and E respectively. Do you think these criteria identify the best models, or are they under-smoothed? Based on the GoS criteria in Figure 5, which models would you recommend?


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**Spatial Smoothing**

**Goodness-of-Fit Statistics**

**Goodness-of-Smoothing Statistics**

**Comparing Spatial Models**

**Key References**